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On a Nonlinear Volterra Integral Equation in a Banach Space

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The equation $u(t) + \int_0^t k(t-s)\, g(s)\, ds \ni f(t)$, $t \geqslant 0$, is studied in a real Banach space with uniformly convex dual. Conditions, sufficient for the existence of a unique solution, are given for the operatorvalued kernel k, the nonlinear m-accretive operators $g(t)$ and the function f. The case when k is realvalued, $g(t) \equiv g$ and X a reflexive Banach space is also considered. These results extend earlier results by Barbu, Londen and MacCamy.

1. INTRODUCTION

We consider existence and uniqueness of solutions of the equation

$$u(t) + \int_0^t k(t-s)\, g(s)\, u(s)\, ds \ni f(t), \qquad t \geqslant 0 \tag{1.1}$$

where k, g and f are given and u is the unknown function taking values in a real Banach space X with uniformly convex dual X^*. The kernel k maps $R_+ = [0, \infty)$ into $L(X)$ (the Banach space of bounded linear operators on X) and $g(s)$ is a nonlinear (in general multivalued) mapping with domain $Dg(s)$ and range $Rg(s)$ in X for a.e. $s \geqslant 0$. We will also consider the case when X is only assumed to be reflexive, $g(s) \equiv g$ and k is a realvalued function.

The essential assumption on the nonlinear mappings $g(s)$ is that they should be m-accretive (see [1, p. 71] for definition). By imposing enough smoothness on the kernel k we are then able to deduce results concerning existence and uniqueness of solutions of (1.1).

Under the hypothesis that X is a real Hilbert space the equation (1.1) has been studied in [4] and under the additional assumption $g(s) \equiv g$ e.g. in [1], [6] and [7]. If $k(t) \equiv I$ (the identity operator) then (1.1) is equivalent to the evolution equation ("$'$" $= d/dt$)

$$\begin{aligned} u'(t) + g(t)\, u(t) &\ni f'(t), \qquad t \geqslant 0 \\ u(0) &= f(0). \end{aligned} \tag{1.2}$$

This equation has been extensively studied in Hilbert and Banach spaces. See [1] and [3] for further references.

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In the proof of Theorem 1 we use the standard method of constructing the solution of (1.1) as the limit of solutions of approximating equations. But the proof relies in an essential way on the fact that under suitable conditions (1.1) is equivalent to a linearly perturbed form of (1.2), see e.g. (3.9) below. In the proof of Theorem 2 we use this fact directly, constructing a solution of this equivalent equation by an iteration procedure. The method of transforming (1.1) into a linearly perturbed form of (1.2) has also been used by MacCamy in [7], but not in the same way as here.

We let $|\ |$ denote the norm in X, X^* and $L(X)$. By $g_\lambda(t)$ is denoted the Yosida approximation of $g(t)$ (i.e. $g_\lambda(t) = (1/\lambda)(I - J_\lambda(t))$ where $J_\lambda(t) = (I + \lambda g(t))^{-1}$; the m-accretivity guanrantees that $DJ_\lambda(t) = X$ for all $\lambda > 0$). We state our first result in

THEOREM 1. *Assume*

$$X \text{ is a real Banach space with uniformly convex dual } X^*, \tag{1.3}$$

$$k \in W^{1,1}_{\text{loc}}(R_+; L(X)), \tag{1.4}$$

$$k' \text{ is locally of bounded variation on } R_+, \tag{1.5}$$

$$k(0) = k_0 I, \qquad k_0 > 0, \tag{1.6}$$

$$f \in W^{1,1}_{\text{loc}}(R_+; X), \tag{1.7}$$

$$f' \text{ is locally of bounded variation on } R_+ \tag{1.8}$$

$$\begin{gathered}\text{there exists a measurable set } E \subset R_+ \text{ such that } m\{R_+ \backslash E\} = 0,\\ 0 \in E \text{ and } t \in E \text{ implies that } g(t) \text{ is an } m\text{-accretive operator in } X,\end{gathered} \tag{1.9}$$

$$\begin{gathered}\text{there exists } \lambda_0 > 0 \text{ such that}\\ |g_\lambda(t)x - g_\lambda(s)x| \leqslant |e(t) - e(s)|\,(L(|x|) + |g_\lambda(s)x|)\\ \forall x \in X, \qquad \forall \lambda \in (0, \lambda_0), \qquad \forall s, t \in E, \qquad s \leqslant t\\ \text{where} \quad e\colon R_+ \to R \quad \text{is such that for some} \quad T_1 > 0\\ \operatorname{Var}(e; [t, t + T_1]) \leqslant q < 1, \qquad \forall t \geqslant 0 \quad \text{and} \quad L\colon R^+ \to R^+\\ \text{is a continuous function,}\end{gathered} \tag{1.10}$$

$$f(0) \in Dg(0). \tag{1.11}$$

Then there exists a unique function $u\colon R_+ \to X$ *such that*

$$u \text{ is locally Lipschitz continuous on } R_+, \tag{1.12}$$

there exists a function $w\colon R_+ \to X$ *such that*

$$w \in L^\infty_{\text{loc}}(R_+; X), \tag{1.13}$$

$$u(t) \in Dg(t), \qquad w(t) \in g(t)\, u(t) \qquad \text{a.e.} \qquad t \geqslant 0 \tag{1.14}$$

and

$$u(t) + \int_0^t k(t-s)\, w(s)\, ds = f(t), \qquad \forall t \geqslant 0. \tag{1.15}$$

Theorem 1 is an extension of [4, Th. 2] to the case when the dual of X is uniformly convex. There it was also assumed that the function L appearing in (1.10) is an affine function. Concerning the kernel k it is to be observed that the key condition is (1.6), the other assumptions are just technical ones. We give next a very simple example of a time-dependent operator that satisfies (1.9) and (1.10): Let g be an m-accretive operator in X and G a function: $X \times R_+ \to X$ such that for any $t \geqslant 0$, $G(\cdot, t)$ is accretive and Lipschitz continuous. Assume moreover that there exists a constant C such that $|G(u, t) - G(u, s)| \leqslant C\,|t - s|\,|u|$. Then it is not difficult to see that the operator $g(t) = g + G(\cdot, t)$ satisfies (1.9) and (1.10), (for the proof of (1.10) see [3, Lemma 3.2]). Our second result is

THEOREM 2. *Assume* (1.7), (1.8) *and*

$$X \text{ is a reflexive real Banach space}, \tag{1.16}$$

$$k \in W^{1,1}_{\text{loc}}(R_+; R), \tag{1.17}$$

$$k' \text{ is locally of bounded variation on } R_+, \tag{1.18}$$

$$k(0) > 0, \tag{1.19}$$

$$g \text{ is an } m\text{-accretive operator in } X, \tag{1.20}$$

$$f(0) \in Dg. \tag{1.21}$$

Then there exists a unique function $u\colon R_+ \to X$ *such that*

$$u \text{ is locally Lipschitz continuous on } R_+, \tag{1.22}$$

there exists a function $w\colon R_+ \to X$ *such that*

$$w \in L^\infty_{\text{loc}}(R_+; X), \tag{1.23}$$

$$u(t) \in Dg, \qquad w(t) \in gu(t), \qquad \text{a.e.} \qquad t \geqslant 0 \tag{1.24}$$

and

$$u(t) + \int_0^t k(t-s)\, w(s)\, ds = f(t), \qquad \forall t \geqslant 0. \tag{1.25}$$

Theorem 2 is an extension of [4, Th. 1] to the case when X is a reflexive Banach space. It is to be observed that the only assumption on g is that it should be m-accretive. In [1], [6] and [7] where the same equation was studied in a Hilbert space other assumptions on the nonlinear operator g were also needed. But on the other hand with less assumptions on g stronger assumptions on the kernel k are needed.

2. Proof of Theorem 1

We consider the approximating equation

$$u_\lambda(t) + \int_0^t k(t-s)\, g_\lambda(s)\, u_\lambda(s)\, ds = f(t), \qquad t \geqslant 0 \tag{2.1}$$

where $\lambda \in (0, \lambda_0)$ and $g_\lambda(s)$ is the Yosida approximation of $g(s)$ when $s \in E$. We observe in the same way as in the proof of [4, Lemma 1] that $Dg(s)$ is independent of s $(= Dg)$ and that $v \in L^p(0, T; X)$ implies $g_\lambda(t)\, v(t) \in L^p(0, T; X)$ for any $p \in [1, \infty]$ and $T > 0$. Using this fact and the Lipschitz continuity (for fixed s) of $g_\lambda(s)$ we conclude from (1.4) and (1.7) that the equation (2.1) has a unique solution u_λ which is locally absolutely continuous on R_+ and such that

$$G_\lambda(t) \overset{\text{def}}{=} g_\lambda(t)\, u_\lambda(t) \in L^\infty_{\text{loc}}(R_+; X).$$

We are now going to construct a solution of (1.1) on the interval $[0, T]$ where $0 < T \leqslant T_1$ and T satisfies

$$k_0 - \| k' \|_{L^1(0,T;L(X))} > 0, \tag{2.2}$$

$$k_0 q + \operatorname{Var}(k'; [0, T])\, T + | k'(0)|\, T < k_0 - \| k' \|_{L^1(0,T;L(X))}, \tag{2.3}$$

$$T(| k_1 | + k_T) \leqslant 1 \tag{2.4}$$

where

$$k_1 = \frac{1}{k_0} k'(0) \tag{2.5}$$

and

$$k_T = (k_0 - \| k' \|_{L^1(0,T;L(X))})^{-1} (\operatorname{Var}(k'; [0, T]) + | k_1 |\, \| k' \|_{L^1(0,T;L(X))}). \tag{2.6}$$

The next step is to deduce some apriori bounds on u_λ, $\dot{u}_\lambda$ and G_λ. But before we proceed with this we need a lemma that enables us to transform (2.1) into a more suitable form. Consequently we will now prove

LEMMA 2.1. *If* (1.3)–(1.6) *hold then there exist sequences* $\{r_n\}$, $\{\epsilon_n\} \subset L^1(0, T; L(X))$ *such that*

$$\| r_n \|_{L^1(0,T;L(X))} \leqslant k_T, \qquad \forall n \in N, \tag{2.7}$$

$$\epsilon_n \to 0 \quad in \quad L^1(0, T; L(X)) \qquad when \qquad n \to \infty, \tag{2.8}$$

$$k'(t) = k_1 k(t) + \int_0^t r_n(s)\, k(t-s)\, ds + \epsilon_n(t) \qquad \text{a.e.} \qquad t \in [0, T]. \tag{2.9}$$

Proof. By (1.4) and (1.5) we can construct a sequence $\{a_n\} \subset W^{1,1}(0, T; L(X))$ with the properties

$$a_n \to k' \quad \text{in} \quad L^1(0, T; L(X)) \qquad \text{when} \qquad n \to \infty, \quad a_n(0) = k'(0), \tag{2.10}$$

$$\| a_n' \|_{L^1(0,T;L(X))} \leqslant \operatorname{Var}(k'; [0, T]), \qquad \forall n \in N. \tag{2.11}$$

(Without loss of generality we assume that k' is continuous from the right at 0). Consider the equation for r_n:

$$k_0 r_n(t) = \int_0^t r_n(s)\, k'(t-s)\, ds = a_n'(t) - \frac{1}{k_0} k'(0)\, k'(t) \qquad \text{a.e.} \qquad t \in [0, T]. \tag{2.12}$$

Using (2.2) and the fixed-point property of contraction mappings we see that (2.12) for each n has a unique solution $r_n \in L^1(0, T; L(X))$. Integrate (2.12) over $(0, t)$. Then it follows by (1.6) and $a_n(0) = k'(0)$ that

$$\int_0^t r_n(s)\, k(t-s)\, ds = a_n(t) - k_1 k(t), \qquad t \in [0, T].$$

Setting $\epsilon_n = k' - a_n$ we now have (2.9). The relations (2.7) and (2.8) are easy consequences of (2.10)–(2.12).

We proceed as suggested in [5] and deduce first an apriori bound on u_λ in

LEMMA 2.2. *If* (1.3)–(1.9) *hold then*

$$\sup_{\lambda \in (0, \lambda_0)} \| u_\lambda \|_{L^\infty(0,T;X)} < \infty. \tag{2.13}$$

Proof. By (1.4) and (1.7) we can differentiate (2.1) to obtain

$$u_\lambda'(t) + k_0 G_\lambda(t) + \int_0^t k'(t-s)\, G_\lambda(s)\, ds = f'(t) \qquad \text{a.e.} \qquad t \in [0, T]. \tag{2.14}$$

It follows from (2.1), (2.9), (2.14) and the associativity of the convolution product that

$$\begin{aligned} u_\lambda'(t) + k_0 G_\lambda(t) - k_1 u_\lambda(t) - \int_0^t r_n(s)\, u_\lambda(t-s)\, ds + \int_0^t \epsilon_n(t-s)\, G_\lambda(s)\, ds \\ = f_n(t) \qquad \text{a.e.} \qquad t \in [0, T], \end{aligned} \tag{2.15}$$

where

$$f_n(t) = f'(t) - k_1 f(t) - \int_0^t r_n(s) f(t-s)\, ds. \tag{2.16}$$

This transformation of (2.14) into (2.15) is the key to the whole proof of Theorem 1.

Define $\langle u, v\rangle = (u, F(v))$ where $(\ ,\ )$ is the duality pairing between X and X^* and $F: X \to X^*$ is the duality mapping $F(x) = \{x^* \in X^* \mid (x, x^*) = |x|^2 = |x^*|^2\}$. As X^* is uniformly convex F is singlevalued. A set $A \subset X \times X$ is by definition accretive if $\langle y_1 - y_2, x_1 - x_2\rangle \geqslant 0\ \forall [x_i, y_i] \in A,\ i = 1, 2$. Let $x \in Dg$. It follows from the absolute continuity of u_λ that $\frac{1}{2}(d/dt) \mid u_\lambda(t) - x\mid^2 = \langle u_\lambda'(t), u_\lambda(t) - x\rangle$ a.e. $t \in [0, T]$, (see [1, p. 100]). The relation (2.15) now implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \mid u_\lambda(t) - x\mid^2 &= -k_0\langle g_\lambda(t)\, u_\lambda(t) - g_\lambda(t)\, x, u_\lambda(t) - x\rangle \\ &\Big\langle -k_0 g_\lambda(t)\, x + k_1 u_\lambda(t) + \int_0^t r_n(s)\, u_\lambda(t-s)\, ds \\ &- \int_0^t \epsilon_n(t-s)\, G_\lambda(s)\, ds + f_n(t), u_\lambda(t) - x\Big\rangle \qquad \text{a.e.} \qquad t \in [0, T].\end{aligned} \tag{2.17}$$

By (1.9) $g_\lambda(t)$ is accretive (in fact m-accretive) for a.e. $t \in [0, T]$ so if we integrate (2.17) over $(0, t)$ and use a quadratic form of Gronwall's lemma (see [2, Lemma A5]) we get

$$\begin{aligned}\mid u_\lambda(t) - x\mid &\leqslant \mid u_\lambda(0) - x\mid + k_0 \int_0^t \mid g_\lambda(s)\, x\mid ds \\ &+ \Big(\mid k_1\mid T + \int_0^T \mid r_n(s)\mid ds T\Big) \| u_\lambda \|_{L^\infty(0,T:X)} \\ &+ T\int_0^T \mid \epsilon_n(s)\mid ds \,\| G_\lambda \|_{L^\infty(0,T:X)} + \| f_n \|_{L^1(0,T:X)} \qquad t \in [0, T].\end{aligned} \tag{2.18}$$

The assumptions $x \in Dg$ and (1.10) imply that

$$\sup_{\lambda \in (0,\lambda_0)} \int_0^T \mid g_\lambda(s)\, x\mid ds < \infty. \tag{2.19}$$

From (2.7) and (2.16) we obtain

$$\sup_n \| f_n \|_{L^1(0,T;X)} < \infty. \tag{2.20}$$

As $u_\lambda(0) = f(0)$ the conclusion of the lemma follows from (2.4)–(2.8) and (2.18)–(2.20).

The next lemma is essential for the rest of the proof as it gives an apriori bound on G_λ.

LEMMA 2.3. *If the assumptions of Theorem 1 hold then*

$$\sup_{\lambda\in(0,\lambda_0)} \| G_\lambda \|_{L^\infty(0,T;X)} \overset{\text{def}}{=} c_T < \infty. \tag{2.21}$$

Proof. Using (1.9) and (2.14) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} | u_\lambda(t+h) - u_\lambda(t)|^2 \\
&\quad = \langle u_\lambda'(t+h) - u_\lambda'(t), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&\quad = -k_0\langle g_\lambda(t+h)\, u_\lambda(t+h) - g_\lambda(t)\, u_\lambda(t), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&\qquad - \left\langle \int_0^{t+h} k'(t+h-s)\, G_\lambda(s)\, ds - \int_0^t k'(t-s)\, G_\lambda(s)\, ds \right. \\
&\qquad \left. - f'(t+h) - f'(t), u_\lambda(t+h) - u_\lambda(t)\right\rangle \\
&\quad \leqslant \Big(k_0 \,| g_\lambda(t+h)\, u_\lambda(t) - g_\lambda(t)\, u_\lambda(t)| \\
&\qquad + \left| \int_0^{t+h} k'(t+h-s)\, G_\lambda(s)\, ds - \int_0^t k'(t-s)\, G_\lambda(s)\, ds \right| \\
&\qquad + |f'(t+h) - f'(t)|\Big)\, | u_\lambda(t+h) - u_\lambda(t)| \qquad \text{a.e.} \qquad t \in [0, T-h].
\end{aligned} \tag{2.22}$$

We can proceed in the same way as in the proof of [4, Lemma 2] and use (2.3) and (2.13) to obtain (2.21).

We now have the necessary tools for proving convergence of u_λ when $\lambda \to 0$. when $\lambda \to 0$. This will be our next task. By (2.15) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} | u_\lambda(t) - u_\mu(t)|^2 \\
&\quad = \langle u_\lambda'(t) - u_\mu'(t), u_\lambda(t) - u_\mu(t)\rangle \\
&\quad = -k_0\langle g_\lambda(t)\, u_\lambda(t) - g_\mu(t)\, u_\mu(t), u_\lambda(t) - u_\mu(t)\rangle \\
&\qquad + \left\langle k_1(u_\lambda(t) - u_\mu(t)) + \int_0^t r_n(s)\, (u_\lambda(t-s) - u_\mu(t-s))\, ds \right. \\
&\qquad \left. - \int_0^t \epsilon_n(t-s)\, (G_\lambda(s) - G_\mu(s))\, ds, u_\lambda(t) - u_\mu(t) \right\rangle dt \qquad \text{a.e.} \qquad t \in [0, T].
\end{aligned} \tag{2.23}$$

As $g_\lambda(t) \in g(t)\, J_\lambda(t)\, u_\lambda(t)$ when $t \in E$ (see [1, p. 73]) and $g(t)$ is accretive we have for a.e. $t \in [0, T]$

$$\begin{aligned}
&-k_0\langle g_\lambda(t)\, u_\lambda(t) - g_\mu(t)\, u_\mu(t), u_\lambda(t) - u_\mu(t)\rangle \\
&\quad \leqslant k_0 \,| G_\lambda(t) - G_\mu(t)|\, |F(u_\lambda(t) - u_\mu(t)) - F(J_\lambda(t)\, u_\lambda(t) - J_\mu(t)\, u_\mu(t))|\,.
\end{aligned} \tag{2.24}$$

Integrating (2.23) over $(0, t)$ and using (2.24) and a quadratic form of Gronwall's lemma ([2, Lemma A5]) we conclude that

$$
\begin{aligned}
&| u_\lambda(t) - u_\mu(t)| \\
&\quad \leqslant \left(2k_0 \int_0^T | G_\lambda(s) - G_\mu(s)| \, |F(u_\lambda(s) - u_\mu(s)) - F(J_\lambda(s)\, u_\lambda(s) - J_\mu(s)\, u_\mu(s))| \, ds\right)^{1/2} \\
&\qquad + | k_1 | \int_0^t | u_\lambda(s) - u_\mu(s)| \, ds + \int_0^t \left| \int_0^s r_n(v) \, (u_\lambda(s-v) - u_\mu(s-v) \, dv \right| ds \\
&\qquad + \int_0^t \left| \int_0^s \epsilon_n(s-v) \, (G_\lambda(v) - G_\mu(v)) \, dv \right| ds, \qquad t \in [0, T].
\end{aligned} \tag{2.25}
$$

Using (2.7) and (2.21) this inequality yields

$$
\begin{aligned}
&\sup_{t\in[0,T]} | u_\lambda(t) - u_\mu(t)| \\
&\quad \leqslant \left(4k_0 c_T \int_0^T | F(u_\lambda(s) - u_\mu(s)) - F(J_\lambda(s)\, u_\lambda(s) - J_\mu(s)\, u_\mu(s))| \, ds\right)^{1/2} \\
&\qquad + 2_T c_T \| \epsilon_n \|_{L^1(0,T;L(X))} + T(| k_1 | + k_T) \sup_{t\in[0,T]} | u_\lambda(t) - u_\mu(t)| \, .
\end{aligned} \tag{2.26}
$$

The uniform convexity of X^* implies that the duality mapping F is uniformly continuous on bounded subsets of X (see [1, p. 14]). By definition $u_\lambda(t) - J_\lambda(t)\, u_\lambda(t) = \lambda g_\lambda(t)\, u_\lambda(t)$ when $t \in E$ so using (2.21), the uniform continuity of F, the dominated convergence theorem, (2.4), (2.8) and (2.26) we see that

$$
\lim_{\lambda,\mu\to 0} \sup_{t\in[0,T]} | u_\lambda(t) - u_\mu(t)| = 0. \tag{2.27}
$$

From (2.27) we deduce the existence of a function $u \in L^2(0, T; X)$ such that

$$
u_\lambda \to u \quad \text{in} \quad L^2(0, T; X) \qquad \text{when} \qquad \lambda \to 0. \tag{2.28}
$$

It follows from (2.21) and the weak sequential compactness of $L^2(0, T; X)$ that there exists a function $w \in L^2(0, T; X)$ such that for some subsequence $\{\lambda_n\}$

$$
G_{\lambda_n} \to w \text{ (weakly)} \quad \text{in} \quad L^2(0, T; X) \qquad \text{when} \qquad \lambda_n \to 0. \tag{2.29}
$$

It is not difficult to see that the mapping $\hat{g} \subset L^2(0, T; X) \times L^2(0, T; X)$ defined by $[p, q] \in \hat{g}$ iff $[p(t), q(t)] \in \hat{g}(t)$ for a.e. $t \in [0, T]$, is m-accretive in $L^2(0, T; x)$, (compare [4, Lemma 4]). As the duality mapping $\bar{F}$: $L^2(0, T; X) \to L^2(0, T; X^*)$ is continuous (because F: $X \to X^*$ is continuous) $\hat{g}$ is demiclosed, which by (2.28) and (2.29) implies that

$$
u(t) \in Dg, \qquad w(t) \in g(t)\, u(t) \qquad \text{for a.e. } t \in [0, T]. \tag{2.30}
$$

By (2.1), (2.28) and (2.29) the equation in (2.31) holds for a.e. $t \in [0, T]$. But as the right side is absolutely continuous by (1.4) and (1.7) we see that it holds for all $t \in [0, T]$, i.e.

$$u(t) = f(t) - \int_0^t k(t-s)\, w(s)\, ds, \qquad t \in [0, T]. \tag{2.31}$$

From (2.21) and (2.29) we conclude that

$$w \in L^\infty(0, T; X) \tag{2.32}$$

So differentiating (2.31) and using (1.3), (1.8) and (2.32) it follows that

$$u \text{ is Lipschitz continuous on } [0, T]. \tag{2.33}$$

Suppose we have another pair of functions u_1, w_1 that satisfies (2.30)–(2.33). If we differentiate (2.31), use (2.9) and the accretivity of $g(t)$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} | u(t) - u_1(t)| &\\ = \langle u'(t) - u_1'(t), u(t) - u_1(t)\rangle &\\ \leqslant \Big\langle k_1(u(t) - u_1(t)) + \int_0^t r_n(s)\,(u(t-s) - u_1(t-s))\, ds &\\ - \int_0^t \epsilon_n(t-s)\,(w(s) - w_1(s))\, ds, u(t) - u_1(t)\Big\rangle &\qquad \text{a.e.} \qquad t \in [0, T]. \end{aligned} \tag{2.34}$$

An integration of (2.31) over $(0, t)$ and an application of an quadratic form of Gronwall's lemma ([2, Lemma A5]) combined with (2.4)–(2.8), (2.31) and (2.32) show that $u(t) = u_1(t)$ for all $t \in [0, T]$.

The only thing left to prove in Theorem 1 is that the solution u can be continued to $[0, \infty)$. This is easily achieved in a standard manner when we observe that we can without loss of generality assume that $u(T) \in Dg$.

3. Proof of Theorem 2

As in the proof of Theorem 1 we begin by proving the existence and uniqueness of a solution on an interval $[0, T]$ where $T > 0$ satisfies

$$k(0) - \| k' \|_{L^1(0,T;R)} > 0 \tag{3.1}$$

and

$$T(| k_1 | + 2k_T) < 1 \tag{3.2}$$

where

$$k_1 = \frac{k'(0)}{k(0)}$$

and (3.3)

$$k_T = (k(0) - \| k' \|_{L^1(0,T;R)})^{-1} (\mathrm{Var}(k'; [0, T]) + | k_1 | \| k' \|_{L^1(0,T;R)}).$$

First we prove a lemma that enables us to study a linearly perturbed differential equation instead of the integral equation (1.1).

LEMMA 3.1. *If* (1.17)–(1.19) *hold then there exists a real measure r of bounded variation on* $[0, T]$ *such that* ($\| \ \|_{[0,T]}$ *denotes the total variation norm*)

$$\| r \|_{[0,T]} \leqslant k_T \tag{3.4}$$

$$\int_0^t k(t-s)\, dr(s) = k'(t) - k_1 k(t) \qquad \text{a.e.} \qquad t \in [0, T]. \tag{3.5}$$

Proof. We proceed exactly as in the proof of Lemma 2.1 and construct a sequence of functions $a_n \in W^{1,1}_{(0,T;R)}$ such that

$$a_n \to k' \quad \text{in} \quad L^1(0, T; R) \qquad \text{when} \qquad n \to \infty \tag{3.6}$$

and

$$\| a_n' \|_{L^1(0,T;R)} \leqslant \mathrm{Var}(k'; [0, T])$$

and (3.7)

$$\| a_n - k' \|_{[0,T]} \to 0 \qquad \text{when} \qquad n \to \infty.$$

We can easily find functions $e_n \in L^1(0, T; R)$, $r_n(t) = \int_0^t e_n(s)\, ds$, such that

$$k(0)\, e_n(t) + \int_0^t k'(t-s)\, e_n(s)\, ds = a_n'(t) - k_1 k'(t) \qquad \text{a.e.} \qquad t \in [0, T]. \tag{3.8}$$

From (3.1), (3.7) and (3.8) we see that there exists a measure r such that (3.4) holds and $\| r_n - r \|_{[0,T]} \to 0$ when $n \to \infty$. An integration of (3.8) yields (3.5) when we let $n \to \infty$.

Our next purpose is to prove the existence and uniqueness of a solution of the equation

$$u'(t) + k(0)\, gu(t) - k_1 u(t) - \int_0^t u(t-s)\, dr(s) \ni f_1(t), \qquad t \in [0, T]. \tag{3.9}$$

When we have done this we show how (1.1) reduces to (3.9) and complete the proof. We state this auxiliary result in

LEMMA 3.2. *If the assumptions of Theorem 2 hold and f_1: $[0, T] \to X$ is of bounded variation, then there exists a unique function u: $[0, T] \to X$ such that*

$$u \text{ is Lipschitz continuous, } u(0) = f(0), \tag{3.10}$$

$$\text{there exists a function } w \in L^\infty(0, T; X) \tag{3.11}$$

$$u(t) \in Dg(t), \qquad w(t) \in gu(t) \qquad \text{a.e.} \qquad t \in [0, T] \tag{3.12}$$

and

$$u'(t) + k(0)\, w(t) - k_1 u(t) - \int_0^t u(t-s)\, dr(s) = f_1(t) \qquad \text{a.e.} \qquad t \in [0, T]. \tag{3.13}$$

Proof. We consider the iteration procedure

$$u_n'(t) + k(0)\, gu_n(t) \ni f_1(t) + k_1 u_{n-1}(t) + \int_0^t u_{n-1}(t-s)\, dr(s), \qquad \text{a.e.} \qquad t \in [0, T]$$
$$u_n(0) = f(0). \tag{3.14}$$

As starting value we can take $u_0(t) = f(0)$, $t \in [0, T]$. If u_{n-1} is Lipschitz continuous on $[0, T]$ then it is easily seen that the function on the right side in (3.14) is of bounded variation on $[0, T]$. More precisely, we have as a crude estimate

$$\begin{aligned} &\operatorname{Var}\left(k_1 u_{n-1}(t) + \int_0^t u_{n-1}(t-s)\, dr(s); [0, T]\right) \\ &\quad \leqslant T(|k_1| + 2\|r\|_{[0,T]})\|u_{n-1}'\|_{L^\infty(0,T;x)} + |u_{n-1}(0)|\,\|r\|_{[0,T]} \end{aligned} \tag{3.15}$$

It is now possible to apply [1, Th. III, 2.2] (observe that $-g$ is a closed dissipative set in $X \times X$ and that we can choose $C = X$) and conclude with the aid of (3.15) that when u_{n-1} is Lipschitz continuous and $u_{n-1}(0) = f(0)$ there exists a unique function u_n: $[0, T] \to X$ such that (3.14) holds and

$$\begin{aligned} \|u_n'\|_{L^\infty(0,T;x)} \leqslant{}& |-y_0 + f_1(0) + k_1 f(0)| + \operatorname{Var}(f_1\colon [0, T]) \\ &+ T(|k_1| + 2\|r\|_{[0,T]})\|u_{n-1}'\|_{L^\infty(0,T;x)} + |f(0)|\,\|r\|_{[0,T]} \end{aligned} \tag{3.16}$$

where $y_0 \in k(0)\, g(f(0))$. (Here we assumed that $r(\{0\}) = 0$ which follows if k' is continuous from the right at 0). When $\langle a, b\rangle_s$ denotes $\sup_{b^* \in F(b)}(a, b^*)$, where F is the duality mapping $X \to X^*$ the function u_n also satisfies the inequality

$$\begin{aligned} \tfrac{1}{2}\|u_n(t) - x\|^2 \leqslant{}& \tfrac{1}{2}\|u_n(s) - y\|^2 + \int_0^t \Big\langle f_1(v) + k_1 u_{n-1}(v) \\ &+ \int_0^v u_{n-1}(v-p)\, dr(p) - y_1 u_n(v) - x\Big\rangle\, dv \end{aligned} \tag{3.17}$$

$$\text{for each} \qquad [x, y] \in k(0)\, g \quad \text{and} \quad 0 \leqslant s \leqslant t \leqslant T.$$

By the accretivity of g the relation (3.14) yields (see also [1, p. 100])

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,|\,u_{n+1}(t)-u_n(t)|^2 &= \langle u'_{n+1}(t)-u'_n(t), u_{n+1}(t)-u_n(t)\rangle_s \\ &\leqslant \left\langle k_1(u_n(t)-u_{n-1}(t)+\int_0^t (u_n(t-s)-u_{n-1}(t-s))\,dr(s), u_{n+1}(t)-u_n(t)\right\rangle_s \\ &\qquad \text{a.e.} \quad t\in[0,T]. \qquad (3.18)\end{aligned}$$

Integrating (3.18) over $(0, t)$, using a quadratic form of Gronwall's lemma ([2, Lemma A5]), we conclude from (3.2)–(3.4) that the sequence $\{u_n\}$ converges uniformly on $[0, T]$ to a function u.

From (3.2)–(3.4) and (3.16) we are able to deduce that u is Lipschitz continuous on $[0, T]$ and so differentiable a.e. as X is reflexive. By the uppersemicontinuity of $\langle\ ,\ \rangle_s$ and the uniform convergence of u_n we obtain by (3.17) the inequality

$$\begin{aligned}\tfrac{1}{2}\,\|\,u(t)-x\,\|^2 &\leqslant \tfrac{1}{2}\,\|\,u(s)-x\,\|^2 \\ &\quad + \int_0^t \left\langle f_1(v)+k_1u(v)+\int_0^v u(v-p)\,dr(p)-y, u(v)-x\right\rangle_s dv, \\ &\qquad [x,y]\in k(0)\,g, \qquad 0\leqslant s\leqslant t\leqslant T. \qquad (3.19)\end{aligned}$$

We can now proceed exactly as in the proof of [1, Th. III 2.2] and conclude that

$$u(t)\in Dg, \qquad f_1(t)+k_1u(t)+\int_0^t u(t-s)\,dr(s)-u'(t)\in k(0)\,gu(t) \quad \text{a.e.} \quad t\in[0,T]. \qquad (3.20)$$

The uniqueness of the function u is obvious from the accretivity of g. The conclusions of Lemma 3.2 follow directly.

Put $f_1 = f'(t)-k_1f(t)-\int_0^t f(t-s)\,dr(s)$. The function f_1 is by (1.8) clearly of bounded variation on $[0, T]$. By Lemma (3.2) there exist functions u and w that satisfy (3.10)–(3.13). We are going to show that this u is a solution to (1.1). Define h by

$$h(t) = u(t)+\int_0^t k(t-s)\,w(s)\,ds, \qquad t\in[0,T]. \qquad (3.20)$$

Differentiating (3.20) and using (3.5) and (3.20) we get

$$\begin{aligned}h'(t) &= u'(t)+k(0)\,w(t)+k_1(h(t)-u(t)) \\ &\quad + \int_0^t (h(t-s)-u(t-s))\,dr(s) \qquad \text{a.e.} \quad t\in[0,T].\end{aligned} \qquad (3.21)$$

As $h(0) = f(0)$ by (3.20) it is obvious that $h(t) = f(t)$ for all $t \in [0, T]$. The conclusions of Theorem 2 for the interval $[0, T]$ now follow from (3.10)–(3.13) and (3.20). That the function u is unique is clear since if u satisfies (1.22)–(1.25) for $[0, T]$ then it also satisfies (3.10)–(3.13). Thus the only thing left is to prove that u can be continued to R_+ and this is achieved in an obvious manner, see [4].

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